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On the self-torque on an extended classical charged particle

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Abstract. The effect of the radiation reaction on the motion of an extended classical charged particle in a magnetic field is investigated. The expression of the self-torque is obtained and it is shown that there are radiationless self-sustained oscillations but no runaways. An application is made to the Hydrogen atom.

1. Introduction

The evolution of a particle with spin in a magnetic field is frequently assumed to be a completely solved problem, not only in classical physics but also in quantum mechanics, where it is relevant to the understanding of atomic structure. However, this is not at all the case if the effect of the radiation emitted by the particle and the corresponding reaction are taken into account. In another paper (Rañada and Rañada 1979) the corrections to the Larmor precession due to the emission of electromagnetic field were studied in the case of a classical charged particle. Because of the importance of the problem it seems convenient to extend that study in order to include the effect of the form of the particle. This is the purpose of this paper.

In § 2 we study the self-torque on a spherical magnetic top. In § 3 we show that there are self-oscillations but no runaways. In § 4 we obtain the self-torque in the limit of a point charge, finally in § 5 we consider the case of the Hydrogen atom, which we study as a classical system by taking the c-number approximation of the spinor wavefunctions.

2. Expression of the electromagnetic self-torque

Let us consider a spherical magnetic top (Goldstein 1951), by which we mean a rigid body with spherically symmetric mass and charge densities and without magnetic moment when at rest. Let its electric form factor be

$$G(k)=\int \rho(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{kr}}\,\mathrm{d}^3\boldsymbol{r} \tag{1}$$

where ρ is the charge density. If the top is rotating with angular velocity $\boldsymbol{\omega}(t)$ in an exterior magnetic field, the radiation reaction will affect its motion by exerting a self-torque, which we will calculate in this section. The current density is

$$\boldsymbol{j}(\boldsymbol{r},t)=\rho(r)\boldsymbol{\omega}(t)\times\boldsymbol{r}. \tag{2}$$

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Let us consider the electromagnetic field in the Coulomb gauge (Jackson 1966)

$$\Delta\phi = -4\pi\rho \qquad \square A = -(4\pi/c)\boldsymbol{j}. \tag{3}$$

By performing the standard Fourier expansion of (3) (Jackson 1966, Bohm and Weinstein 1948), it is very easy to find the radiated field

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{2}{\pi}\int_0^\infty \mathrm{d}\tau\, \boldsymbol{\omega}(t-\tau)\times\frac{\boldsymbol{r}}{r}\int_0^\infty \mathrm{d}k \left(\frac{\mathrm{d}G}{\mathrm{d}k}\right)\sin ck\tau\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\sin kr}{kr}\right), \tag{4}$$

after which the self-torque can be calculated as

$$\boldsymbol{0}_{\mathrm{self}} = \int \boldsymbol{r}\times(\rho(r)\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{j}(\boldsymbol{r}, t)\times\boldsymbol{B}(\boldsymbol{r}, t))\,\mathrm{d}\boldsymbol{r}$$

with the result

$$\boldsymbol{0}_{\mathrm{self}} = \int_0^\infty \left[\boldsymbol{\omega}(t)\times\boldsymbol{\omega}(t-\tau) - \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}(t-\tau)\right]K(\tau)\,\mathrm{d}\tau \tag{5}$$

where

$$K(\tau) = \frac{4}{3\pi c}\int_0^\infty kG'^2 \sin ck\tau\,\mathrm{d}k. \tag{6}$$

As $\boldsymbol{0}_{\mathrm{self}}$ is minus the time derivative of the electromagnetic spin, the evolution equation for the mechanical spin $\boldsymbol{S}$ is

$$\mathrm{d}\boldsymbol{S}/\mathrm{d}t = \boldsymbol{0}_{\mathrm{self}} + \boldsymbol{0}_{\mathrm{ext}} \tag{7}$$

where $\boldsymbol{0}_{\mathrm{ext}}$ is due to the external magnetic field. As our top is classical we can write $\boldsymbol{S} = I\boldsymbol{\omega}$, where I is the mechanical moment of inertia.

It should be stressed that, as far as the spherical charge is rigid, (5) is an exact consequence of electrodynamics, presupposing no restriction on either angular velocity or acceleration.

3. Self-oscillations and non-existence of runaways

A curious property of equation (7) is the existence of oscillatory solutions, even in the absence of an external torque. They represent harmonic motions in which the direction of $\boldsymbol{\omega}$ remains constant, while its modulus varies sinusoidally with time, the energy going back and forth between the top and the electromagnetic field. Since the work done over a period is zero, these oscillations are self-sustained. Similar motions were considered by Sommerfeld (1905) in a spherical shell model of the electron (Schott 1933, Erber 1961).

Inserting $\boldsymbol{\omega} = \boldsymbol{\omega}_0\,\mathrm{e}^{-\mathrm{i}\lambda t}$ in (7) we obtain the conditions for the existence of these self-oscillations

$$\mathrm{d}G/\mathrm{d}k = 0, \qquad k = \lambda/c \tag{9a}$$

$$I = \frac{4}{3\pi}\int_0^\infty \mathrm{d}k\frac{ck^2}{\lambda^2 - c^2k^2}\left(\frac{\mathrm{d}G}{\mathrm{d}k}\right)^2. \tag{9b}$$

The spectrum of frequencies at which the system can oscillate without radiation thus depends strongly on the shape of the charge. As an example, let us consider the case of a spherical shell which has density $\rho=(e/4\pi a^2)\delta(r-a)$. In this case $G(k)=e(\sin ka)/ka$. The condition (9*a*) for self-oscillation is

$$\tan \lambda a/c=\lambda a/c$$

and the solutions are possible only when $I=0$.

An attractive feature of equation (7) is the absence of runaway solutions of the form $\boldsymbol{S}=\boldsymbol{S}_0\,\mathrm{e}^{\lambda t}$, λ real, which often appear in problems of this kind. By inserting this expression in (7) we obtain

$$I=-\frac{4}{3\pi}\int_0^\infty \mathrm{d}k\frac{ck^2}{\lambda^2+c^2k^2}\left(\frac{\mathrm{d}G}{\mathrm{d}k}\right)^2 \tag{10}$$

which is impossible since the mechanical moment of inertia I cannot be negative.

4. The self-torque in the limit of a point charge

Let us consider the behaviour of $\mathbf{0}_{\text{self}}$ when the radius a of the charge goes to zero. In order to do this we develop (5) in series as follows

$$\mathbf{0}_{\text{self}}=\sum_{n=0}^{\infty} A_n(\boldsymbol{\omega}(t)\times\boldsymbol{\omega}^{(n)}(t)-\boldsymbol{\omega}^{(n+1)}(t)) \tag{11}$$

where

$$A_n=\frac{(-1)^n}{n!}\int_0^\infty K(\tau)\tau^n\,\mathrm{d}\tau. \tag{12}$$

Since we are considering spherical distributions of charge of radius a

$$\rho(r)=0 \qquad r>a. \tag{13}$$

The expansion (11) is well defined because in this case

$$K(\tau)=0 \qquad \text{for} \qquad \tau>2a/c. \tag{14}$$

This can be checked by considering the definition of $K(\tau)$.

The coefficients A_n depend on what charge distribution $\rho(r)$ we have, this is also reflected in the limit $a\to 0$. We consider two cases as a manifestation of the non-uniqueness of the point-like limit.

(a) *Spherical shell*

$$\rho_{\mathrm{e}}(r)=(e/4\pi a^2)\delta(r-a) \qquad G(k)=e(\sin ka)/ka$$

$$K(\tau)=\begin{cases}(e^2/6c)(2-c^2\tau^2/a^2) & \tau<2a/c\\ 0 & \tau>2a/c.\end{cases}$$

In this way

$$A_n=\frac{(-1)^n2^{n+1}}{3n!}\,\frac{(1-n)}{(n+1)(n+3)}\,\frac{e^2a^{n+1}}{c^{n+2}}. \tag{15}$$

If we suppose a mass density $\rho_{\mathrm{m}}(r)=(m/4\pi a^2)\delta(r-a)$. The corresponding mechanical moment of inertia is $I=\frac{2}{3}ma^2$.

(b) *Uniform sphere*

$$\rho_r(r)=\begin{cases}3e/4\pi a^3 & r\leqslant a\\ 0 & r>a\end{cases}\qquad G(k)=[3e/(ka)^3](\sin ka-ka\cos ka)$$

$$K(\tau)=(e^2/3c)(c\tau/a)[\tfrac{2}{5}-\tfrac{1}{2}c\tau/a+\tfrac{1}{8}(c\tau/a)^3-\tfrac{1}{80}(c\tau/a)^5].$$

We obtain

$$A_n=\frac{(-1)^n2^{n+3}}{3n!}\frac{(1-n)}{(n+2)(n+3)(n+5)(n+7)}\frac{e^2a^{n+1}}{c^{n+2}}. \tag{16}$$

Considering a mass density

$$\rho_m(r)=\begin{cases}3m/4\pi a^3 & r\leqslant a\\ 0 & r>a,\end{cases}$$

the corresponding mechanical moment of inertia is

$$I=\tfrac{2}{5}ma^2.$$

By introducing the spin in equation (11), we get

$$\mathbf{O}_{\text{self}}=\sum_{n=0}^{\infty}\frac{A_n}{I^2}\boldsymbol{S}(t)\times\boldsymbol{S}^{(n)}(t)-\sum_{n=0}^{\infty}\frac{A_n}{I}\boldsymbol{S}^{(n+1)}(t). \tag{17}$$

With the above expressions we obtain, when $a\to 0$,
Spherical shell:

$$\mathbf{O}_{\text{self}}=-\frac{1}{a}\left(\frac{1}{3}\frac{e^2}{mc^2}\dot{\boldsymbol{S}}+\frac{1}{5}\frac{e^2}{m^2c^4}\boldsymbol{S}\times\ddot{\boldsymbol{S}}\right)+\frac{1}{6}\frac{e^2}{m^2c^5}\boldsymbol{S}\times\dddot{\boldsymbol{S}}+\mathrm{O}(a) \tag{18}$$

Uniform sphere:

$$\mathbf{O}_{\text{self}}=-\frac{1}{a}\left(\frac{2}{7}\frac{e^2}{mc^2}\dot{\boldsymbol{S}}+\frac{5}{21}\frac{e^2}{m^2c^4}\boldsymbol{S}\times\ddot{\boldsymbol{S}}\right)+\frac{1}{6}\frac{e^2}{m^2c^5}\boldsymbol{S}\times\dddot{\boldsymbol{S}}+\mathrm{O}(a). \tag{19}$$

In both cases the linear terms for $n>0$ and the non-linear terms for $n>3$ are zero when $a\to 0$. On the other hand there are two terms which become infinite ($\sim a^{-1}$) while a non-linear term is independent of a. That means we have to consider three terms for charges which are highly localised.

The above expressions of $\mathbf{O}_{\text{self}}$ are different from those obtained by Rañada and Rañada (1979). The reason is the following. In the 1979 paper the self-torque was obtained as the flux of the angular momentum density tensor through a sphere of radius $r\to\infty$. It included, therefore, the contribution of the near zone and was not identical to the self-torque on the mechanical part of the particle. The fact that equation (7) does not admit runaway solutions suggests that their existence is related to the inclusion of terms which are due to storage of angular momentum in the near zone. This is a very interesting question which will be discussed in another paper.

5. An application to the Hydrogen atom

In the preceding sections we have considered the electromagnetic self-torque in the case of an extended rigid charge. We will now apply the same ideas to the Hydrogen atom. This may seem strange, since it is clearly a very different system. However, as a first approximation we can treat the Dirac wavefunctions as c-number fields (Rañada 1977, Rañada and Usón 1980, 1981) and study the evolution of the corresponding charge and magnetic moment densities. Although we are thus considering a classical field theory, the results and relation which can be obtained may be useful in the study of the problems which appear in the correct quantum theory.

Let us consider a Hydrogen atom in its ground state. If $s_z = \frac{1}{2}\hbar$, the spinor wavefunction is (Bethe and Salpeter 1957)

$$\psi = \mathrm{e}^{-\mathrm{i}\omega t} \frac{1}{\sqrt{4\pi}} \begin{pmatrix} g\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ -\mathrm{i}f\begin{pmatrix} \cos\theta \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} \end{pmatrix}. \tag{20}$$

If the spin is directed along the unit vector $\boldsymbol{n} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$, the charge and current densities are

$$\rho = (e/\hbar c)\psi^{\dagger}\psi = (e/4\pi\hbar c)(f^2 + g^2) \tag{21a}$$

$$\boldsymbol{j} = (e/\hbar)\bar{\psi}\boldsymbol{\gamma}\psi = -(e/4\pi\hbar)(2fg/r)\boldsymbol{n}\times\boldsymbol{r}. \tag{21b}$$

We assume that, if the atom is submitted to an external magnetic field, the angles α and β vary but its internal structure is not modified or, in other words, that it behaves as a rigid body. This is clearly a very safe assumption, except for very large values of B. Proceeding as in § 2, we can deduce from (3) and (21) the value of the self-torque, which turns out to be

$$\boldsymbol{0}_{\text{self}} = \int_0^{\infty} \left(\boldsymbol{n}(t)\times\boldsymbol{n}(t-\tau)A(\tau) - \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}(t-\tau)B(\tau) \right) \mathrm{d}\tau \tag{22}$$

where

$$A(\tau) = \frac{4}{3\pi c}\int_0^{\infty} \mathrm{d}k\, kH'^2 \sin ck\tau, \qquad B(\tau) = \frac{4}{3\pi c}\int_0^{\infty} \mathrm{d}k\, kH'G' \sin ck\tau \tag{23}$$

G being given as before by (1) and H being the magnetic form factor

$$H(k) = \int \frac{-e}{4\pi\hbar}\frac{2gf}{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r}. \tag{24}$$

In order to estimate A and B, we may use the non-relativistic approximation

$$g = 2(\hbar c/r_0^3)^{1/2}\,\mathrm{e}^{-r/r_0}, \qquad f \simeq +\tfrac{1}{2}\alpha g \tag{25}$$

where α is the fine structure constant and r_0 the Bohr radius. We thus obtain

$$A(\tau) = (\alpha^2 e^2 c/36 r_0^2)\nu(\nu^2 + 3\nu - 69)\mathrm{e}^{-\nu}$$

$$B(\tau) = (1 + \tfrac{1}{4}\alpha^2)(\alpha e^2/16 r_0)(8 - 15\nu + 15\nu^2 - \tfrac{2}{3}\nu^3 + \tfrac{1}{9}\nu^4)\mathrm{e}^{-\nu} \tag{26}$$

where $\nu = 2c\tau/r_0$. The main difference between (5) and (22) is that the latter makes use, instead of only one, of two different delay functions related to the electric and magnetic effects. It must be noted that if the charge occupies a finite volume and can be contained in a sphere of radius R, the functions A and B vanish for $\tau > 2R/c$. In general they decrease at infinity.

It must also be pointed out that (22) has neither self-oscillations nor runaway solutions.

Let us finally mention that it is possible to develop (22) in series as

$$\mathbf{0}_{\text{self}} = \sum_{0}^{\infty} (A_k \boldsymbol{n}(t) \times \boldsymbol{n}^k(t) - B_k \boldsymbol{n}^{k+1}(t)) \tag{27}$$

where

$$A_k = \frac{(-1)^k}{k!} \int_0^{\infty} A(\tau)\tau^k \, \mathrm{d}\tau \qquad B_k = \frac{-(-1)^k}{k!} \int_0^{\infty} B(\tau)\tau^k \, \mathrm{d}\tau,$$

$\boldsymbol{n}^{(k)}$ is the kth derivative of $\boldsymbol{n}$ and the expansion parameter is a time which characterises the average delay in (7).

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